Minimal step number of knots with small crossing number

— 交点数の小さい結び目の最小ステップ数について —

Saitama University JSPS Fellow Kai Ishihara [1]

ある結び目を lattice knot で構成するのに必要な長さを minimal step number という。Diao [1] によって trefoil knot の minimal step number が 24 であることが示された。また、山口氏 [3] によって figure eight knot の minimal step number が 30 であることが示された。ここでは、5_1 knot の minimal step number が 34 であることを示す。

1 Introduction

The *steps* are unit segments with endpoints in Z^3 in R^3. A simple closed polygonal cycle consisting of steps is called a *lattice knot*. The *minimal step number* is the number of steps required to form a lattice knot. The trefoil knot can be constructed by 24 steps. So the minimal step number of the trefoil knot is at most 24. Diao showed the following theorem.

Theorem 1 ([1]). *A lattice knot with a step number less than* 24 *is an unknot. Therefore, the minimal step number of the trefoil knot is* 24.

The figure eight knot can be constructed by 30 steps. So the minimal step number of the figure eight knot is at most 30. Yamaguchi showed the following theorem.

Theorem 2 ([3]). *A lattice knot with a step number less than* 30 *is an unknot or a trefoil knot. Hence, the minimal step number of the figure eight knot is* 30.

By constructing a lattice knot, we can give an upper bound for the minimal step number. For knots with small crossing number, the minimal step numbers are estimated by computer simulation [2].

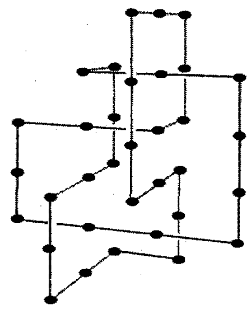
2 Result

We obtain the following theorem.

[1] E-mail: kisihara@rimath.saitama-u.ac.jp

Theorem 3. *A lattice knot with a step number less than* 34 *has crossing number at most* 4*, and so that is an unknot or a trefoil knot or a figure eight knot. Therefore, the minimal step number of the* 5_1 *knot is* 34.

A lattice 5_1 knot in the figure in the right gives the minimal step number.



Lattice 5_1 knot with 34 steps

3 Outline of proof

Let P be a lattice knot giving the minimal step number 32 at most. We will show that the crossing number of P is at most 4. Let G be the projection of P to xy-plane, that is a graph on $\mathbb{R}^2$. Each edge of G is an unit segment and corresponds to x-steps or y-steps of P. Here a step parallel to a-axis ($a \in \{x, y, z\}$) is called an *a-step*. For each edge of G, the number of corresponding x-steps or y-steps is called the *multiplicity*. The sum of multiplicities for all edges of G is called the *total multiplicity*, that is the number of all x-steps and y-steps of P. Assuming the number of z-steps of P is most, the total multiplicity of G is at most 20. We consider all possibilities of such graphs with multiplicities on $\mathbb{R}^2$. This is a same method as Diao [1]. We add some ideas to this. First, there are some conditions for G where P does not give the minimal step number. For example, if the 3×4 region does not contain G, then we can show P does not give the minimal step number, and so it contradicts the first assumption. Hence, it is enough to consider the graphs in the 3×4 region. Next, we can give an upper bound for the crossing number of P by calculation for multiplicities of G. Using such ideas, we can show that the crossing number of P is at most 4.

Detail proof will soon be appear.

参考文献

[1] Y. Diao, Minimal knotted polygons on the cubic lattice. J. Knot Theory Ramifications **2** (1993), no. 4, 413-425.

[2] E. J. Janse van Rensburg and S. D. Promislow, Minimal knots in the cubic lattice. J. Knot Theory Ramifications **4** (1995), no. 1, 115-130.

[3] Y. Yamaguchi, Minimal step number of figure eight knot. Master Thesis, Saitama University, March 2008.